# WHEN NONRANDOMNESS APPEARS RANDOM: A CHALLENGE TO DECISION SCIENCES

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## ABSTRACT

This paper reviews the methodological foundations of deterministic and random modeling and argues that determinism remains the scientific goal of any investigation. We make a contribution by performing an experiment based on the most famous chaotic deterministic system of the Lorenz equations and demonstrate that the currently available techniques for distinguishing between deterministic and random systems are not adequate. The two techniques we employ are the correlation dimension and the BDS test.

## INTRODUCTION

Determinism and randomness are the two pillars of scientific methodology. Ruhla (1992) [4] argues that science, in its long historical evolution, has favored determinism. In other words, the search for an exact relationship between dependent and independent variables has received first priority by scientists who follow the deterministic tradition of Euclid, Newton and Leibnitz. The probabilistic paradigm, which originated in the rigorous analysis of gambling games, has flourished during the past several decades as exact relationships have become more difficult to confirm.

Developments in operations research, management science and economic analysis since World War II have reflected the evolution of scientific methodology in the physical sciences. The Marshallian static equilibrium price theory, the Walrasian dynamic tâtonement general equilibrium, linear programming techniques, game theory and various other techniques, all emphasized classical determinism. However, measurement errors, unobservable variables, incomplete models, the introduction of expectations and the admission of the economic and business complexity, among other reasons, have swung the methodological pendulum towards probabilistic reasoning. The need to forecast an uncertain future variable for purposes of economic and financial planning has reinforced probabilistic methods. Such reasoning gave rise to statistical techniques and the establishment of the field of decision sciences.

Although it is currently accepted by decision scientists that there is a clear dichotomy between deterministic and probabilistic modeling, relatively recent developments in physical chaotic dynamics have shown that certain processes, while they appear to be random, need not in fact be random. It is the purpose of this paper to first review rapidly these ideas and, second to consider a model that is deterministic and ask the fundamental question: "When does nonrandomness appear random?". Put differently, suppose that an exact, deterministic theoretical model is developed between certain variables: when or how can a decision scientist conclude, by observing exact time series measurements of such variables, that these variables are random?

The remainder of the paper is organized as follows. Section 2 briefly contrasts the notion of deterministic and random models while section 3 presents the most famous deterministic system that behaves like a random one, i.e. the Lorenz equations. Our contribution is exposited in section 4 where we sample from the Lorenz equations and posit the question: when or how can a decision scientist uncover whether the model under analysis is deterministic or random.

## DETERMINISTIC VERSUS RANDOM MODELS

Deterministic models consist of exact relationships. Abstracting from specific modeling considerations, the notion of determinism is clearly demonstrated in the relationship of a function:

$$y = f(x) \quad (1)$$

where f denotes the set of ordered pairs (x,y). In other words, each x is unambiguously associated with a specific y, with such a y being equal to f(x). From the simple calculus where $f(x)\colon \; R \to R$, R denoting the real numbers, to multivariate calculus, differential equations, real analysis and functional analysis, the subject matter remains exact relationships between or among certain variables. These exact relationships can become quite complicated, particularly when such a relationship is between derivatives (i.e. differential equations) or even among functions themselves (i.e. functional analysis). Nevertheless, in all instances, such relationships are exact.

From Euclid's geometry, to Newton's calculus and to today's advanced analysis, the subject matter of scientific investigations is determinism. Discovering, establishing, analyzing and understanding exact relationships among certain variables remains today's highest scientific goal, not only of mathematicians, but also of applied researchers, such as physicists and management scientists. Only after such a primary goal has not been reached, do scientists consider second best solutions by studying nondeterministic models. Such models are also called random or stochastic and are mostly substitutes rather than competing alternatives for the deterministic truth.

Mathematics, which one could argue remains the most rigorous of human scientific efforts, demonstrates that, independent of its intrinsic interest, randomness is not an alternative of equal standing but a temporary substitute to determinism. From the elementary probability, where one flips a coin, to measure-theoretic probability, the notion of a function prevails. What changes is the domain of the function. In probability, the domain is a random set and a function that takes its values from a random set is called a random variable. Ruhla (1992) [4] describes with great scientific care the relationship between these two methodologies by arguing that probability is a branch in the scientific tree named determinism.

## THE LORENZ EQUATIONS

Our discussion thus far was carried out at the methodological level. In other words, in searching for causal relationships, a scientist may choose an exact or a random model. We have argued that exactness has been given priority in the applied sciences and in pure mathematics, while randomness is viewed as a temporary methodological substitute. How can we further strengthen our argument towards determinism?

Chaotic dynamics was developed precisely for this purpose: to demonstrate that there exist exact functions which generate very complicated trajectories that appear like random. From the seminal work of Eckman and Ruelle (1985) [2] to the numerous texts about dynamics scientists have exposited an exciting new branch of mathematics which reinforces determinism.

Limitations of space do not allow us to describe in detail the key ideas, definitions and theorems of chaotic dynamics. Here, for the sake of continuity, we give the fundamental definition of chaotic dynamics. We say that a function $f\colon\ R \to R$ is chaotic if it satisfies three conditions: (a) f is topologically transitive, (b) f has sensitive dependence on initial conditions, and (c) f has periodic points that are dense in the real numbers.

The Lorenz (1963) [3] equations are the most famous example of a system that generates chaotic dynamics. They are:

$$x_t = s(-x_{t-1} + y_{t-1}) \quad (2)$$
$$y_t = rx_{t-1} - y_{t-1} - x_{t-1}z_{t-1} \quad (3)$$
$$z_t = -bz_{t-1} + x_{t-1}y_{t-1} \quad (4)$$

This system of equations is represented here by difference equations. They can also be expressed as a system of differential equations, as was initially derived by Lorenz (1963) [3] in his meteorological study of a three-equation approximation to the motion of a layer of fluid heated from below. Observe that there are three parameters, s,r and b. More specifically, the parameter r corresponds to the Reynolds number and as it varies, the system goes through remarkable qualitative changes. For parameter values $b = 2.667$, $r = 28.0$ and $s = 10.0$, almost all solutions converge to a set called the strange attractor. Furthermore, once on the attractor, these solutions exhibit random-like behavior. An exhaustive analysis of the numerous properties of these equations may be found in Sparrow (1982) [5].

## THE EXPERIMENT

Using the deterministic Lorenz equations, with a step size of 0.1, we generated three sets of 5000 observations each. The first set records each value of the variable x generated by the Lorenz equations. In other words, the first set has jump = 1. The jumps of the second and the third set are 10 and 100 respectively. The exact size of these two jumps is not critical; other numbers such as 20 and 50 or 250 and 500, etc., could have been chosen; what we wish to illustrate is three levels of information: all values, every tenth value and every hundredth value, where these three procedures correspond to detailed sampling, frequent sampling, and infrequent sampling. Obviously, to keep the number of observations the same, the interval of the second set is longer than the first and the third is longer than the second.

We next ask the fundamental question: what methods are available to the decision scientist to allow him/her to distinguish whether a data set of observations is generated by a deterministic or random function? Scientists from various backgrounds have researched this question extensively. For our purposes, we use the two main techniques, namely, the correlation dimension and the BDS tests.

For a given $\epsilon > 0$, define the correlation integral, denoted by $C^M(\epsilon)$, to be:

$$C^M(\epsilon) = \frac{\text{the number of pairs}(s,t) \text{ whose distance } \| x^M(s) - x^M(t) \| < \epsilon}{T^2_M}$$

$$= \frac{\text{the number of } (s,t),\ 1 \le t,\ s \le T\ \ \| x^M(s) - x^M(t) \| < \epsilon}{T^2_M} \tag{5}$$

where $T_M = (T + 1) - (M - 1)$, and as before $x^M(t) = [x(t), x(t + 1), \ldots, x(t + M \text{ -m } 1)]$.

Observe that $\| . \|$ in (5) denotes vector norm. Using the correlation integral, we can define the <u>correlation dimension</u> for an embedding dimension M as :

$$D^M = \lim_{\substack{\epsilon \to 0 \\ T \to \infty}} \frac{\ln C^M(\epsilon)}{\ln(\epsilon)} . \tag{6}$$

In (6) ln denotes natural logarithm. Finally, the correlation dimension D is given by:

$$D = \lim_{M \to \infty} D^M . \tag{7}$$

The second test we perform is the BDS, extensively presented in Brock, Hsieh and LeBaron (1991) [1]. These authors report that for an independent and identically distributed

$$C^M(\epsilon, T) \to [C^1(\epsilon)]^M \quad as \quad T \to \infty \tag{8}$$

random process and for fixed M-histories and $\epsilon > 0$

$$\sqrt{T}(C^M(\epsilon,T) - [C^1(\epsilon,T)]^M) \to N \quad (0, \sigma^2(\epsilon,T)) , \tag{9}$$

They further report that as $T$ approaches infinity, where N denotes a normal distribution with mean zero and variance $\sigma^2(\epsilon,T)$. From the above two equations (8) and (9), it is concluded that

$$\frac{\sqrt{T}(C^M(\epsilon,T) - [C^1(\epsilon,T)]^M)}{\sigma^M(\epsilon,T)} \to N(0,1) \tag{10}$$

The correlation dimension performs well when every value of the Lorenz equation is sampled, but does poorly when the jump increases to 10 and then to 100. This illustrates that unless, in the real world, we can record information at high frequencies rather than at prespecified intervals, say end of the day, weekly, monthly, etc., we are bound to lose the underlying structure. Our experiment shows that infrequent sampling misses the deterministic relationship. Of course, data limitations may not allow a scientist to perform the tests we used. For example using annual or quarterly data, one does not have enough observations to do dimension and BDS analysis. The BDS does very well rejecting randomness in our sample, but cannot specify the alternative. This test is new and offers great promise. Again, Brock, Hsieh and LeBaron (1991) [1] report numerous applications for this test that have taken place recently.

## CONCLUSION

Our overall conclusion is simply this: since WWII, the scientific pendulum in general and in management science, operations research and forecasting in particular, has been pulled away from determinism and brought towards stochasticity. But such stochasticity has not fully enriched our understanding of the real world simply because what drives randomness often cannot be anticipated. Chaotic dynamics is not a totally new methodology, but rather a new way of affirming order, rationality and exactness despite the seeming disorderly, illogical and random behavior of certain variables. This discovery of chaotic dynamics and our illustration of the Lorenz equation may hopefully be viewed as an opportunity to push the methodological pendulum back towards determinism.

## REFERENCES


[1] Brock, W., D. Hsieh and B. LeBaron, (1991), *Nonlinear Dynamics, Chaos and Instability: Statistical Theory and Economic Evidence*, Cambridge, Massachusettes: The MIT Press.

[2] Eckmann, J. and D. Ruelle, (1985), "Ergodic Theory of Chaos and Strange Attractors", *Review of Modern Physics*, Vol. 57, pp. 617-656.

[3] Lorenz, E., (1963), "Deterministic Non-Periodic Flows", *Journal of Atmospheric Sciences*, 20, pp. 130-141.

[4] Ruhla, C., (1992), *The Physics of Chance: From Plaise to Niels Bohr*, New York: Oxford University Press.

[5] Sparrow, C., (1982), *The Lorenz Equations: Bifurcations , Chaos, and Strange Attractors*, New York: Springer-Verlag.